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published in
European Journal of Applied Mathematics
2004

DOI (link to publisher)
10.1017/S0956792504005510

document version
Publisher's PDF, also known as Version of record

citation for published version (APA)
van den Berg, G. J. B., Bowen, M., King, J. R., & El-Sheikh, M. M. A. (2004). The self-similar solution for draining in the thin film equation. *European Journal of Applied Mathematics*, *15*(3), 329-346. https://doi.org/10.1017/S0956792504005510

Euro. Jnl of Applied Mathematics (2004), *vol.* 15, *pp.* 329–346.
DOI: 10.1017/S0956792504005510 Printed in the United Kingdom

The self-similar solution for draining in the thin film equation

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(*Received 8 July 2002; revised 7 October 2003*)

We investigate self-similar solutions of the thin film equation in the case of zero contact angle boundary conditions on a finite domain. We prove existence and uniqueness of such a solution and determine the asymptotic behaviour as the exponent in the equation approaches the critical value at which zero contact angle boundary conditions become untenable. Numerical and power-series solutions are also presented.

1 Introduction

We consider the thin film equation on a bounded domain with zero contact angle boundary conditions

$$\begin{cases} h_t = -(h^n h_{xxx})_x, & t > 0,\ x \in (-L, L) \\ h(\pm L) = h_x(\pm L) = 0, & h > 0 \text{ on } (-L, L). \end{cases} \tag{1.1}$$

Here $n \in \mathbb{R}$ is a parameter; certain restrictions on the range of n are expected to apply, as explained below. The thin film equation has the scaling invariance $(h, x, t) \mapsto (\mu h, \lambda x, \mu^n \lambda^4 t)$ with $\mu, \lambda > 0$. In this paper we investigate scaling invariant solutions of (1.1), which in the present context of a fixed bounded domain are separable ($\lambda = 1$).

The thin film equation arises in a number of contexts, such as in the description of droplets of viscous fluid spreading under surface tension. We refer to numerous authors [1, 2, 6, 7, 8, 14, 16] for overviews and extensive lists of references. The zero contact angle boundary conditions in (1.1) have been chosen to reflect a situation where fluid is either draining over an edge or is absorbed by a porous substrate outside of the domain $(-L, L)$; see Bowen *et al.* [9], for example. Problem (1.1) may also describe aspects of draining through singularities or near-singularities which can arise due to film rupture [5].

There may be mathematical difficulties in imposing a zero contact angle on the boundary. This is well known in the context of moving contact lines, where for $n \geqslant 3$ the solution cannot spread beyond its initial support. The situation for fixed boundary conditions

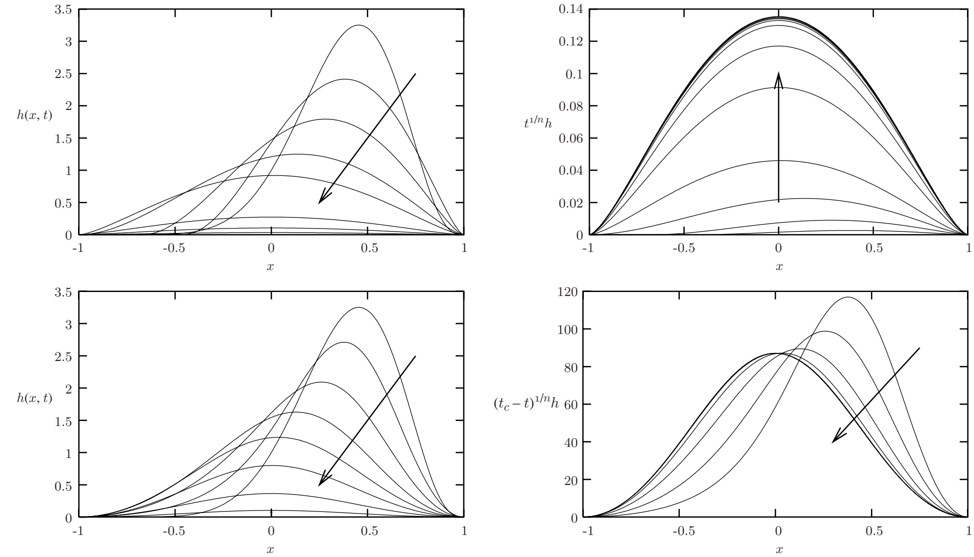

FIGURE 1. The evolution of solutions of (1.1) for $n = 1$ (top) and $n = -1$ (bottom). The arrows indicate the direction in which t increases. The left pictures show the decay to zero in physical coordinates. On the right the vertical axis is scaled appropriately to illustrate convergence to a separable solution as $t \to \infty$ ($n = 1$) or $t \to t_c$ ($n = -1$).

has been studied less widely than the moving boundary case. Formal analysis of the asymptotic behaviour of solutions near the boundary [3, 9, 10] suggests that for $n \geqslant 2$ the flux $h^n h_{xxx}$ at the boundary is always inwards and has to be prescribed. This does not make much sense in the context of draining and it clearly falls outside the scope of (1.1). For $n < 2$ it is found that outward flux is a genuine possibility. Although (for $n < \frac{3}{2}$) the support may initially retract from the fixed boundary, it eventually covers the whole interval and draining occurs.

Existence theory for the initial value problem of the thin film equation was developed in Bernis & Friedman [2]. It does not, however, cover the zero contact angle boundary conditions studied in the present paper. We do not consider the existence problem directly, but we show via ordinary differential equation methods that the thin film equation has particular solutions satisfying the boundary conditions provided $n < 2$. We note that this stands in contrast to the porous medium equation $h_t = (h^m h_x)_x$ (the corresponding *second* order parabolic equation with scaling invariance) with Dirichlet boundary data, which has nontrivial solutions for $m > -1$ only.

The scaling invariance suggests that when a meaningful solution of (1.1) exists, its large time (or extinction) behaviour is governed by a separable solution. This expectation is confirmed by numerical calculations. In Figure 1 we have depicted the evolution of solutions for $n = 1$ and $n = -1$, both in physical coordinates (where they decay to zero) and in scaled coordinates (where they converge to a nontrivial profile). Notice that, to illustrate the universal nature of the (symmetric) asymptotic solution, the solutions start from asymmetric initial data. These numerical results strongly support the idea that the large time behaviour of (1.1) is self-similar; see also Bowen & King [10].

The study of separable solutions not only contributes to the description of this asymptotic behaviour, but it also gives further evidence for the criticality of $n = 2$ for zero contact angle conditions on fixed boundaries. Moreover, rigorous analysis of separable solution provides another step towards understanding the set of self-similar solutions of the thin film equation (cf. Bernis *et al.* [4] for source type solutions and Bernis *et al.* [3] and Bowen *et al.* [9] for dipole solutions). We note that the methods in this paper are genuinely different from those previously used for source type and dipole solutions. The method developed here incorporates the determination of both lower and upper bounds on the solution, allowing a rigorous check of formal results obtained from matched asymptotics. This will be illustrated by examining the limit behaviour of the self-similar solutions as $n \uparrow 2$.

Investigation of self-similar solutions for higher-order equations, such as the thin film equation, is a challenging task because the analysis of the second order equation $h_t = (h^m h_x)_x$ relies heavily on maximum principle arguments and phase plane analysis, neither of which is available for higher-order equations. We note that the separable case has a special status in the sense that the resulting ordinary differential equation has a translational symmetry as well as a scaling one (so it can be reduced to a second order non-autonomous (or third order autonomous) equation, but this is not the most practical way to proceed, as will become clear below).

Looking for separable solutions, we write $h = (1+t)^{-1/n} v(x)$ and obtain infinite time extinction for $n > 0$ with v satisfying

$$(v^n v''')' = \frac{1}{n} v. \tag{1.2}$$

For $n < 0$ we use the Ansatz $h = (1-t)^{-1/n} v(x)$, which extinguishes in finite time, to arrive at

$$(v^n v''')' = \frac{1}{|n|} v.$$

Finally, for the borderline value $n = 0$ the equation is linear and an explicit solution can be constructed (which has infinite time extinction). To capture all cases in one equation, we use the scaling $u = |n|^{1/n} v$ and obtain

$$(u^n u''')' = u. \tag{1.3}$$

Instead of working on a fixed domain $(-L, L)$ we exploit the scaling invariance to convert to solutions of fixed maximum height, while *a priori* the interval length $2L_n$ is unknown:

$$\begin{cases} (u^n u''')' = u, \\ u > 0 \text{ on } (-L_n, L_n), \\ u(\pm L_n) = u'(\pm L_n) = 0, \\ \max_{x \in (-L_n, L_n)} u(x) = 1. \end{cases} \tag{1.4}$$

We have thus used the scaling invariance to fix the height of the solution; the remaining, translational symmetry will be used to reduce the order of the equation (see § 2).

As observed in Bernis *et al.* [3] and Bowen & King [10], the boundary conditions require the constraint $n < 2$. In particular, it follows rigorously from arguments analogous to Bernis *et al.* [3, § 6] that *no* solution of (1.4) exists for $n \geqslant 2$, implying that draining via zero contact angle is not possible for $n \geqslant 2$. In that case physically meaningful solutions of (1.1) conserve mass and have a non-zero contact angle at the edges of the support; defining

$$J = -\, u^n u'''|_{x=-L} \tag{1.5}$$

to be the flux of material leaving the domain $|x| < L$ through $x = -L$ we must therefore replace the zero contact angle boundary condition at $x = -L$ with $J = 0$ (a similar zero flux condition must also be imposed at $x = L$) and the asymptotic behaviour is given by a steady-state solution (a parabola) rather than by a separable one. When n approaches 2 from below the separable solution approaches a parabola, while the flux through the boundary tends to zero in appropriately chosen coordinates (the asymptotic behaviour is given by (1.8) and (3.3)).

It is interesting, and quite helpful in the proof of our main results, to consider the asymptotic behaviour near the boundaries $x = \pm L_n$ under the assumption that the flux across the boundaries is nonzero (as it must be when draining occurs). Keeping in mind that we are looking for solutions with zero contact angle, a straightforward formal computation gives

$$u \sim \begin{cases} C(x \pm L_n)^2 & n < \frac{1}{2} \\ C(x \pm L_n)^2 |\ln(x \pm L_n)|^{2/3} & n = \frac{1}{2} \\ C(x \pm L_n)^{3/(n+1)} & n > \frac{1}{2} \end{cases} \qquad \text{with } C > 0. \tag{1.6}$$

A rigorous proof follows directly via arguments similar to those provided in Bernis *et al.* [3, § 6]. The formula also shows how regularity is lost as $n \uparrow 2$. For $n \geqslant 2$ this asymptotic behaviour is not compatible with the zero contact angle boundary conditions, which explains the nonexistence result for $n \geqslant 2$.

Returning to the separable solutions, with the nonexistence result for $n \geqslant 2$ in mind we restrict our attention to the parameter regime $n < 2$.

Theorem 1 *Problem* (1.4) *has a unique solution* (u_n, L_n) *for any* $n < 2$. *The solution is symmetric, i.e.* $u_n(-x) = u_n(x)$, *and depends continuously on* n. *As* $n \uparrow 2$ *the asymptotic behaviour is* $u_n(x) = 1 - \frac{1}{2}(2-n)^{-1/2}x^2 + O(2-n)$ *and* $L_n = 2^{1/2}(2-n)^{1/4} + O((2-n)^{5/4})$.

The following is a direct consequence of the scaling invariance and Theorem 1:

Corollary 2 *Let* (u_n, L_n) *be the solutions defined in Theorem* 1. *All separable solution of* (1.1) *are given by*

$$h = \begin{cases} (L/L_n)^{4/n}|n|^{-1/n}\,(T-t)^{1/|n|}\,u_n\!\left(\frac{L_n}{L}x\right) & n < 0 \\ e^{-(L_0/L)^4(t+T)}\,u_0\!\left(\frac{L_0}{L}x\right) & n = 0 \\ (L/L_n)^{4/n}n^{-1/n}\,(T+t)^{-1/n}\,u_n\!\left(\frac{L_n}{L}x\right) & 0 < n < 2 \end{cases} \qquad \textit{with } T \in \mathbb{R}.$$

The asymptotic behaviour as $n \uparrow 2$ can be compared with the formal results from Bowen & Kin. Looking at solutions of (1.2) on the domain $(-1, 1)$ one finds after

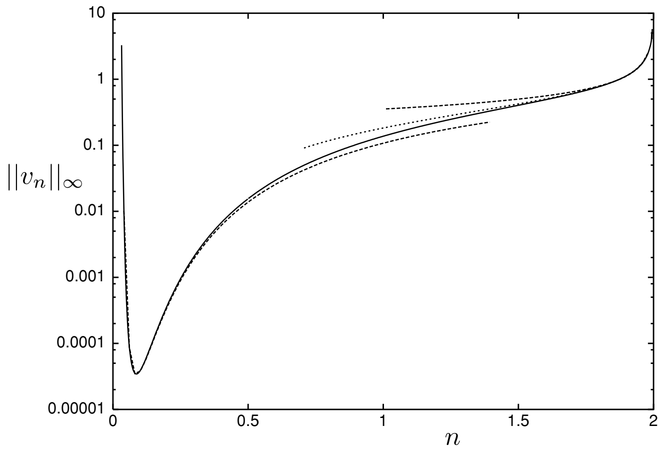

FIGURE 2. A comparison of numerically calculated solutions (solid line) with the asymptotics (equations (1.7) and (1.9); dashed lines) and the higher order approximation (1.8) (dotted line) for $0<n<2$.

appropriately rescaling the result in Theorem 1 that asymptotically

$$v_n \sim 2^{-3/2}(2-n)^{-1/2}(1-x^2) \qquad \text{as} \quad n \uparrow 2, \tag{1.7}$$

which indeed was predicted in [10]. In Figure 2 we compare the asymptotic formula from Theorem 1 with numerical results. Higher-order correction terms to the result (1.7) can be fairly readily constructed formally, yielding

$$\|v_n\|_\infty \sim \frac{1}{\sqrt{8\varepsilon}}\left(1-\frac{1}{4}\varepsilon\ln\varepsilon-\frac{\varepsilon}{36}(7+15\ln 2)\right) \qquad \text{as} \quad n \uparrow 2, \tag{1.8}$$

where $\varepsilon = 2-n$, and this more accurate approximation is also shown in Figure 2.

In addition, it was predicted in Bowen & King [10], again by application of formal asymptotics, that

$$\|v_n\|_\infty \sim \alpha\left(n\lambda_0^4\right)^{-1/n}(\cosh\lambda_0-\cos\lambda_0) \qquad \text{as} \quad n \downarrow 0 \tag{1.9}$$

where λ_0 is the smallest positive root of $\tanh\lambda + \tan\lambda = 0$ and $\ln(\alpha) = \int_{-1}^{1} f_0^2(1-\ln f_0)\,dx / \int_{-1}^{1} f_0^2\,dx$, where $f_0(x) = \cosh(\lambda_0)\cos(\lambda_0 x) - \cos(\lambda_0)\cosh(\lambda_0 x)$ is the zeroth eigenmode for $n=0$. Figure 2 also demonstrates strong numerical support of this result. The leading order behaviour $\ln(||v_n||_\infty) \sim -(\ln n)/n - (4\ln\lambda_0)/n$ as $n \downarrow 0$ follows from the continuity result in Theorem 1. The value of the multiplicative constant in (1.9) could be obtained rigorously by constructing bounds suited to the limit $n \to 0$. In this paper we only perform such a detailed analysis for the limit $n \uparrow 2$, which we find more interesting from the point of criticality (besides, it may serve as an example for other limits such as $n \to 0$ and $n \to -\infty$, cf. Bowen & King [10] for formal results).

In § 2 we present a proof of Theorem 1. The numerical method is described in § 3 and an alternative (semi-analytic) method for constructing solutions in § 4. We make some concluding comments in § 5.

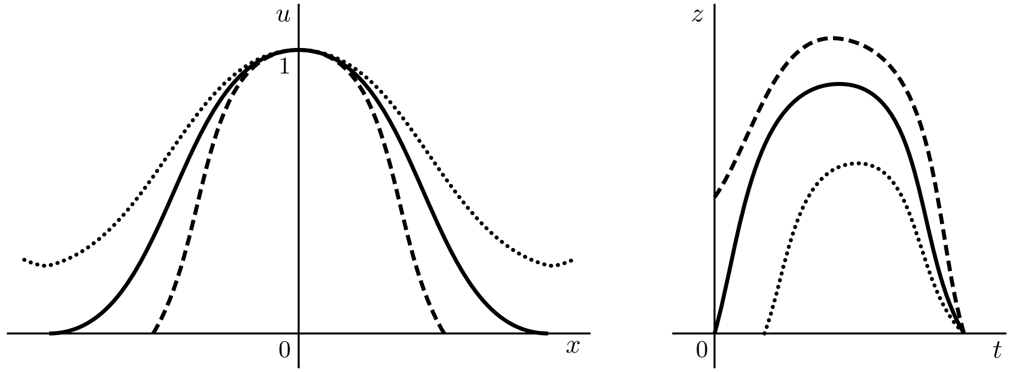

FIGURE 3. A sketch of three symmetric profiles and their corresponding representations in z-t coordinates.

2 Proof of Theorem 1

Let us briefly outline the idea of the proof. We first establish that any separable solution is symmetric. To prove existence of a solution we then consider the ordinary differential equation $(u^n u''')' = u$ with initial values $(u, u', u'', u''')(0) = (1, 0, u''(0), 0)$, where we use $u''(0)$ as a "shooting parameter". For large negative $u''(0)$ the solution hits zero with nonzero derivative, whereas for small negative $u''(0)$ the solution does not reach zero. Intuitively one expects to find a solution that touches zero with zero slope for some value of $u''(0)$ "in between". To make these statements rigorous (the last one in particular) we will introduce supersolutions and subsolutions. These will also play a role in proving uniqueness.

We start with some preliminary observations. Equation (1.3) implies that u''' has at most one sign change (from negative to positive). It easily follows from the boundary conditions that any solution of (1.4) has precisely two monotone laps, and that at the maximum $x_0 \in (-L_n, L_n)$ one has $u(x_0) = 1$, $u'(x_0) = 0$, $u''(x_0) < 0$.

On monotone laps we can invert the role of the dependent and independent variables and write $x(u)$. Introducing $t = u \geqslant 0$ and $z(t) = 2^{-1/2} u'^2$, i.e. $z(u) = 2^{-1/2}[\frac{du}{dx}(x(u))]^2$, we obtain for the two monotone laps of u:

$$\begin{cases} \sqrt{z}\dfrac{d}{dt}t^n\sqrt{z}\dfrac{d^2}{dt^2}z = t, \\ z(0) = z(1) = 0 \text{ and } z > 0 \text{ on } (0,1). \end{cases} \tag{2.1}$$

The change of variables from $u(x)$ to $u'(u)$ on monotone laps is a natural one. It exploits the translation invariance of the fourth order autonomous problem to convert it into a third order non-autonomous problem. One could in fact use the scaling invariance to go one step further and obtain a (complicated) second order non-autonomous equation, but this did not turn out to be useful in our approach. The choice to define z as the square of u' was made to keep expressions as simple as possible, while the new explicit independent variable $t = u$ was introduced to distinguish clearly between the roles of u as dependent and independent variable. The two coordinate systems are depicted in Figure 3.

Throughout we shall use primes to denote both derivatives of u with respect to x and derivatives of z with respect to t; it should be clear from the context which one is meant.

Notice that $z' = 2^{1/2}u''$ and $\sqrt{z}z'' = 2^{1/4}u'''$. A warning seems relevant: in the last formula we interpret $\sqrt{z}$ as a $2^{-1/4}u'$, but throughout we will use the positive root, so it is best to always think about increasing laps (possibly after applying a reflection). In the following we will write for convenience

$$\mathscr{L}z(t) \stackrel{\text{def}}{=} t^n\sqrt{z(t)}z''(t),$$

which corresponds to the flux in u coordinates. In the boundary points (where the equation degenerates) the derivatives of z may be unbounded, but $\mathscr{L}z$ should be finite.

In (2.1) we have a third order equation with only two boundary conditions. On the other hand, a solution of (1.4) has two monotone laps, hence in principle we need to find two solution of (2.1) with additional conditions that guarantee the corresponding solution of (1.4) is sufficiently smooth at its maximum. However, as we show below, a solution of (1.4) is symmetric, which corresponds to one solution of (2.1) with the additional boundary condition $\mathscr{L}z(1) = 0$, thus leading to a third order problem with three boundary conditions.

Although (2.1) is a third order equation, we use some notions that appear frequently in the context of *second* order equations. First we introduce the notion of super- and subsolutions.

Definition 3 *We call y a supersolution if $\sqrt{y}(\mathscr{L}y)' \geqslant t$ and subsolution if $\sqrt{y}(\mathscr{L}y)' \leqslant t$.*

Comparison properties are common in second order equations, and they usually provide information about uniqueness and qualitative behaviour. Although our equation is not of second order, we obtain a crucial *comparison lemma* which serves similar purposes. The lemma implies that under suitable conditions (which are satisfied in the circumstances we encounter later) subsolutions lie below supersolutions.

Lemma 4 *Let y_1 be a supersolution and y_2 a subsolution on (t_0, t_1). Let $y_1(t_0) = y_2(t_0)$, $y_1(t_1) = y_2(t_1)$ and $\mathscr{L}y_1(t_1) \leqslant \mathscr{L}y_2(t_1)$ and $\mathscr{L}y_1(t_1) \leqslant 0$. Suppose that $0 < y_1 \leqslant y_2$ on (t_0, t_1). Then $y_1 \equiv y_2$.*

Proof First, $\mathscr{L}y_1 \leqslant \mathscr{L}y_2$ on (t_0, t_1):

$$\begin{aligned}\mathscr{L}y_1(t) &= \mathscr{L}y_1(t_1) - \int_t^{t_1} (\mathscr{L}y_1)'(s)\,ds \leqslant \mathscr{L}y_1(t_1) - \int_t^{t_1} \frac{s}{\sqrt{y_1(s)}}\,ds \\ &\leqslant \mathscr{L}y_2(t_1) - \int_t^{t_1} \frac{s}{\sqrt{y_2(s)}}\,ds \leqslant \mathscr{L}y_2(t_1) - \int_t^{t_1} (\mathscr{L}y_2)'(s)\,ds = \mathscr{L}y_2(t).\end{aligned} \tag{2.2}$$

Second, from (2.2) and the assumptions we see that $\mathscr{L}y_1 < 0$ and hence $y_1'' < 0$ on (t_0, t_1). We conclude that $\sqrt{y_2}y_1'' \leqslant \sqrt{y_1}y_1''$, and from $\mathscr{L}y_1 \leqslant \mathscr{L}y_2$ that $\sqrt{y_1}y_1'' \leqslant \sqrt{y_2}y_2''$ and hence $y_1'' \leqslant y_2''$ on (t_0, t_1). Together with the assumptions $y_1(t_0) = y_2(t_0)$, $y_1(t_1) = y_2(t_1)$ and $y_2 \geqslant y_1$ on (t_0, t_1), the maximum principle implies that $y_2 \equiv y_1$. $\square$

Uniqueness follows fairly directly from the previous comparison lemma.

Lemma 5 *There is at most one solution of* (1.4). *It must be symmetric and corresponds to a solution of* (2.1) *with* $\mathscr{L}z(1)=0$.

Proof First, there is no asymmetric solution. Suppose by contradiction that there is one. Then the monotone laps are two different solutions z_1 and z_2 of (2.1). Since $u'''(x_0)\neq 0$ for an asymmetric solution we will have (possibly after exchanging indices) $\mathscr{L}z_1(1)<0<\mathscr{L}z_2(1)$ and $z_1<z_2$ on $(1-\delta,1)$ for small $\delta>0$. Define $t_0=\inf\{0<t<1\,|\,z_1<z_2 \text{ on } (t,1)\}$. Then $z_1(t_0)=z_2(t_0)$ and Lemma 4 proves that $z_1\equiv z_2$ on $[t_0,1]$, a contradiction.

Second, there is at most one symmetric solution. Suppose there are two symmetric solutions with corresponding monotone laps z_1 and z_2. Since $u_1'''(0)=u_2'''(0)=0$ and $u_1''(0)\neq u_2''(0)$, it follows (possibly after exchanging indices) that $\mathscr{L}z_1(1)=\mathscr{L}z_2(1)=0$ and $z_1'(1)<z_2'(1)$, hence $z_1>z_2$ on $(1-\delta,1)$ for small $\delta>0$. The same reasoning as before leads to a contradiction. □

In the theory of second order elliptic equations the existence of an ordered pair of a subsolution and a supersolution often implies the existence of a solution in between. For our third order problem we require additional constraints on the pair, which is why we introduce the notion of supersub pairs. A supersolution y_1 and a subsolution y_2 form a supersub pair if they satisfy the *mixed* constraints

$$\sqrt{y_2}(\mathscr{L}y_1)'>t \qquad \text{and} \qquad \sqrt{y_1}(\mathscr{L}y_2)'<t,$$

as well as several technical requirements. To be precise:

Definition 6 *We call* (y_1,y_2) *a* supersub pair *if* $\sqrt{y_2}(\mathscr{L}y_1)'>t$ *and* $\sqrt{y_1}(\mathscr{L}y_2)'<t$ *on* $(0,1)$, $y_1(0)=y_2(0)=y_1(1)=y_2(1)=0$, $y_1>y_2$ *on* $(0,1)$ *and* $\mathscr{L}y_1(1)\leqslant\mathscr{L}y_2(1)=0$ *and* $y_1'(1)\leqslant y_2'(1)<0$.

It follows from this definition that when (y_1,y_2) is a supersub pair then y_1 is a supersolution and y_2 is a subsolution. Furthermore, if (y_1,y_2) is a supersub pair, then so is $(\lambda y_1,\lambda^{-1}y_2)$ for all $\lambda>1$.

While we have borrowed concept from elliptic partial differential equations, our interpretation is in terms of orbits of an ordinary differential equation. In particular, we have some control over solutions which lie between a supersub pair. Indeed, supersub pairs have the following squeezing property:

Lemma 7 *Let* z *be a solution of* $\sqrt{z}(\mathscr{L}z)'=t$ *on* $(t_0,1)$ *with* $z(t_0)=z(1)=0$. *Let* (y_1,y_2) *be a supersub pair. Suppose* $\mathscr{L}y_1(1)\leqslant\mathscr{L}z(1)\leqslant\mathscr{L}y_2(1)=0$ *and* $y_2<z<y_1$ *on* $(1-\delta,1)$ *for some* $\delta>0$. *Then* $z\leqslant y_1$ *on* $(t_0,1]$. *If* $t_0=0$ *then* $y_2\leqslant z\leqslant y_1$ *on* $[0,1]$.

Proof First notice that y_1 is a supersolution and y_2 is a subsolution. Let

$$t_1=\inf\{t\in(t_0,1)\,|\,y_2\leqslant z\leqslant y_1 \text{ on } (t,1]\}.$$

If $t_1=t_0$ we are done. If $t_1>t_0$ then there are two possibilities: $y_1(t_1)=z(t_1)$ or $y_2(t_1)=z(t_1)$. We will first exclude the former. Suppose that $y_1(t_1)=z(t_1)$ so $z>y_1$ in a left neighbourhood of t_1 (this follows from the (first) strict inequality in Definition 6). Let

$$t_2 = \inf\{t < t_1 \,|\, z > y_1 \text{ on } (t, t_1)\}.$$

Clearly, $z > y_1$ on (t_2, t_1) and $z(t_2) = y(t_2)$, $z(t_1) = y(t_1)$. Also,

$$\begin{aligned}\mathscr{L}z(t_1) = \mathscr{L}z(1) - \int_{t_1}^{1} \frac{s}{\sqrt{z(s)}}\,ds &\geqslant \mathscr{L}y_1(1) - \int_{t_1}^{1} \frac{s}{\sqrt{y_2(s)}}\,ds \\ &\geqslant \mathscr{L}y_1(1) - \int_{t_1}^{1} (\mathscr{L}y_1)'(s)\,ds = \mathscr{L}y_1(t_1).\end{aligned}$$

Lemma 4 gives a contradiction.

We are left with the case $y_2(t_1) = z(t_1)$. Let

$$t_3 = \inf\{t < t_1 \,|\, z < y_2 \text{ on } (t, t_1)\}.$$

If $t_3 = t_0 > 0$ then we are done. If $t_3 > t_0$ or if $t_3 = t_0 = 0$ then $z(t_3) = y_2(t_3)$ and $z(t_1) = y(t_1)$, and $z < y_2$ on (t_3, t_1). Also,

$$\begin{aligned}\mathscr{L}z(t_1) = \mathscr{L}z(1) - \int_{t_1}^{1} \frac{s}{\sqrt{z(s)}}\,ds &\leqslant \mathscr{L}y_2(1) - \int_{t_1}^{1} \frac{s}{\sqrt{y_1(s)}}\,ds \\ &\leqslant \mathscr{L}y_2(1) - \int_{t_1}^{1} (\mathscr{L}y_2)'(s)\,ds = \mathscr{L}y_2(t_2).\end{aligned}$$

Lemma 4 now again gives a contradiction, which completes the proof. □

In particular, a supersub pair gives bounds on the solutions of (2.1).

Lemma 8 *Let* (y_1, y_2) *be a supersub pair. If* z *is solution of* (2.1) *with* $\mathscr{L}z(1) = 0$, *then* $y_2 \leqslant z \leqslant y_1$.

Proof If $y_1'(1) < z'(1) < y_2'(1) < 0$ then Lemma 7 gives the result. If $z'(1) < y_1'(1)$ or $z'(1) > y_2'(1)$ then Lemma 4 leads to a contradiction since y_1 is supersolution and y_2 is subsolution. If none of the inequalities hold, i.e. $z'(1) = y_1'(1)$ or $z'(1) = y_2'(1)$, then $\lambda y_1'(1) < z'(1) < \lambda^{-1} y_2'(1)$ for any $\lambda > 1$. Since $(\lambda y_1, \lambda^{-1} y_2)$ is a supersub pair one obtains from Lemma 7 that $\lambda^{-1} y_2 \leqslant z \leqslant \lambda y_1$ and the required result is obtained by taking the limit $\lambda \downarrow 1$. □

Supersub pairs do actually exist but only for $n < 2$. We remark that the shape of the supersub pair is partly determined by the asymptotic behaviour near the boundary (1.6).

Lemma 9 *For* $n < 2$ *there exists a supersub pair. In particular, for* μ *sufficiently large*

$$y_1 = \begin{cases} \mu\big(t^{(4-2n)/3} - t\big) & n > \frac{1}{2} \\ \mu\, t \ln t^{-1} & n = \frac{1}{2} \\ \mu\big(t - t^{(4-2n)/3}\big) & n < \frac{1}{2} \end{cases} \qquad \text{and} \qquad y_2 = \mu^{-1}(t - t^{3-n}) \tag{2.3}$$

form a supersub pair.

Proof For $n \neq \frac{1}{2}$ one finds

$$g_1 \stackrel{\text{def}}{=} t^{-1}\sqrt{y_2}(\mathscr{L}y_1)' = \mu\frac{(2-n)(2n-1)^2}{27}\,t^{(n-5)/3}\sqrt{\frac{t-t^{3-n}}{|t-t^{(4-2n)/3}|}}$$

$$g_2 \stackrel{\text{def}}{=} t^{-1}\sqrt{y_1}(\mathscr{L}y_2)' = \mu^{-1}\frac{(3-n)(2-n)}{2}(-3+(5-n)t^{2-n})\sqrt{\frac{|t-t^{(4-2n)/3}|}{t-t^{3-n}}}$$

We want to show that, for μ sufficiently large, $g_1 > 1 > g_2$ for all $t \in (0,1)$. All other requirements are easily checked. Since $\mu g_2(t)$ is a bounded function on $[0,1]$ and $\mu^{-1}g_1(t)$ is positive and bounded away from 0, one obtains $g_1 > 1 > g_2$ on $(0,1)$ for μ sufficiently large. For $n = \frac{1}{2}$ the same line of arguments holds. $\square$

Having already proved uniqueness we now address the issue of existence.

Lemma 10 *For $n<2$ there exists a solution of* (2.1) *with* $\mathscr{L}z(1)=0$.

Proof Let (y_1, y_2) be the supersub pair from Lemma 9. We are going to apply a shooting argument with $\alpha = z'(1)$ as shooting parameter. Let z_α be the solution with $z(1)=0$, $z'(1) = \alpha$ and $\mathscr{L}z(1)=0$, corresponding to $(u,u',u'',u''')(0) = (1,0,\alpha/\sqrt{2},0)$. If $z_\alpha(t) = 0$ for some $t \in [0,1)$, then we define t_α to be this zero, i.e. $z_\alpha(t_\alpha) = 0$ and $z_\alpha > 0$ on $(t_\alpha, 1)$, and we do not (try to) continue the solution below t_α.

For each $\alpha<0$ either $z_\alpha(t_\alpha)=0$ for some $t_\alpha \in [0,1)$ or $z_\alpha(0) > 0$. It follows from Lemma 4 that if $\alpha > y_2'(1)$ then z_α hits 0 at some point $t_\alpha > 0$ and $z_\alpha < y_2 < y_1$ on $[t_\alpha, 1)$, while if $\alpha < y_1'(1)$ then z_α hits the z-axis and $z_\alpha > 0$ on $[0,1)$. Define

$$\alpha_0 \stackrel{\text{def}}{=} \inf\{\alpha < 0 \mid z_\alpha(t_\alpha) = 0 \text{ for some } t_\alpha \in [0,1)\}.$$

Clearly $\alpha_0 \geqslant y_1'(1)$ is well-defined and we claim that z_{α_0} is the desired solution. First notice that $t_\alpha < t_\beta$ and $z_\alpha > z_\beta$ on $[t_\beta, 1)$ for all $\beta > \alpha > \alpha_0$ by Lemma 4.

Furthermore, t_α is continuous as a function of α as long as $t_\alpha > 0$. In particular, if $t_\alpha > 0$ then $t_\beta \in (0,1)$ is well-defined for β sufficiently close to α. This is most easily seen by looking at the corresponding coordinates in the u phase space: $u = t_\alpha > 0$, $u' = 0$, and it follows from the differential equation that $u'' > 0$ (since the only sign change of u''' is at $u = 1$). Hence by the implicit function theorem and continuity of the solution as a function of α, the minimal value t_α of u is a continuous function of α.

We thus deduce from the definition of α_0 that $t_\alpha \downarrow 0$ as $\alpha \downarrow \alpha_0$. Therefore, $z_\alpha(t) \uparrow z_{\alpha_0}(t)$ as $\alpha \downarrow \alpha_0$ for all $t \in (0,1)$. Finally, $z_\alpha < y_1$ on $[t_\alpha, 1)$ for all $\alpha < \alpha_0$ by Lemma 4. Hence $z_{\alpha_0} \leqslant y_1$ on $(0,1)$ and thus $z_{\alpha_0}(0) = 0$, which completes the proof. $\square$

Having proved existence and uniqueness, we define $z_n(t)$ to be the unique solution of

$$\begin{cases} \sqrt{z_n}(t^n\sqrt{z_n}z_n'')' = t, \\ z_n > 0 \text{ on } (0,1), \\ z_n(0) = z_n(1) = \mathscr{L}z_n(1) = 0. \end{cases} \tag{2.4}$$

The corresponding solution $u_n(x)$ of (1.4) is implicitly defined by

$$x = \pm \int_{u_n}^{1} \frac{2^{-1/4}}{\sqrt{z_n(s)}}\, ds, \qquad u_n \in [0,1].$$

Lemma 11 *The solutions* z_n *of* (2.4) *and* (u_n, L_n) *of* (1.4) *depend continuously on* n *for* $n<2$.

Proof It is not difficult to see from the proof of Lemma 9 that the constant μ can be chosen uniformly for n in compact subsets of $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, 2)$. For n close to $\frac{1}{2}$ it suffices to redefine $y_1 = \frac{3}{2}\frac{\mu}{|n-\frac{1}{2}|}|t - t^{(4-2n)/3}|$, $n \neq \frac{1}{2}$. Then μ can again be chosen uniformly in the neighbourhood of $n = \frac{1}{2}$, and

$$\lim_{n\to\frac{1}{2}} \frac{3}{2}\frac{\mu}{|n-\frac{1}{2}|}\left|t - t^{(4-2n)/3}\right| = -\mu\, t \ln t,$$

so that the transition through $n = \frac{1}{2}$ is smooth. This also explains the choice for $n = \frac{1}{2}$ in (2.3).

With the bounds produced by Lemma 8, which are uniform by the above arguments, it is not difficult to verify that $u_n \to u_{n_0}$ as $n \to n_0 \in (-\infty, 2)$. □

To calculate the asymptotic profile of the solution as $n \uparrow 2$ we need a better supersub pair.

Lemma 12 *Let* $n = 2 - \varepsilon$. *For* $\varepsilon > 0$ *sufficiently small the following is a supersub pair:*

$$y_1 = C_\varepsilon\left(t^{2\varepsilon/3} - t^{1+\varepsilon/3-2\varepsilon^2/3}\right) \qquad \text{and} \qquad y_2 = D_\varepsilon\left(t^{2\varepsilon/3} - t^{1+\varepsilon/3}\right)$$

with

$$C_\varepsilon = \left[\frac{3^7 4}{\varepsilon^2(3+2\varepsilon)^4(1-\varepsilon)^4(3-2\varepsilon)^3}\right]^{1/4} \qquad \text{and} \qquad D_\varepsilon = \left[\frac{3^3 4(3-2\varepsilon)}{\varepsilon^2(3-\varepsilon)^4}\right]^{1/4}.$$

Proof Straightforward but tedious calculations give

$$\begin{aligned} h_1 &\stackrel{\text{def}}{=} \sqrt{y_2}(\mathscr{L}y_1)' \\ &= E_\varepsilon\left[2\varepsilon(1+2\varepsilon)t^{-2\varepsilon^2/3} + 3(1+\varepsilon)(1-2\varepsilon)t^{1-\varepsilon/3-4\varepsilon^2/3}\right]\sqrt{\frac{t^{\varepsilon/3} - t}{t^{\varepsilon/3} - t^{1-2\varepsilon^2/3}}}, \end{aligned}$$

where $E_\varepsilon = \frac{1}{54}C_\varepsilon^{3/2}D_\varepsilon^{1/2}\varepsilon(9-4\varepsilon^2)(1-\varepsilon)$, from which we infer that

$$h_1 = t + \tfrac{1}{3}\varepsilon(-t\ln t - 2t + 2) + \tfrac{2\varepsilon(1+2\varepsilon)}{\sqrt{(3-\varepsilon)(1-\varepsilon)(3+2\varepsilon)}}\left(t^{-2\varepsilon^2/3} - 1\right) + \varepsilon^2 H_1(t,\varepsilon),$$

where $H_1(t,\varepsilon)$ is a uniformly bounded function on $[0,1] \times [0,\varepsilon_0]$ for ε_0 sufficiently small, $H_1(1,\varepsilon) = 0$ and $H_1 \in C^1((0,1] \times [0,\varepsilon_0])$. Since the second and third terms are positive and dominate the remaining term, one obtains that $h_1 > t$ on $(0,1)$ for ε sufficiently small.

Likewise,

$$h_2 \stackrel{\text{def}}{=} \sqrt{y_1}(\mathscr{L}y_2)' = \frac{\varepsilon D_\varepsilon^{3/2} C_\varepsilon^{1/2}}{18}\left[-6\varepsilon + 2\varepsilon^2 + (9-\varepsilon^2)t^{1-\varepsilon/3}\right]\sqrt{\frac{t^{\varepsilon/3} - t^{1-2\varepsilon^2/3}}{t^{\varepsilon/3} - t}},$$

from which we infer that

$$h_2 = t - \tfrac{1}{3}\varepsilon\left(t\ln t - 2t + 2\right) + \varepsilon^2 H_2(t,\varepsilon)$$

where $H_2(t,\varepsilon)$ is a uniformly bounded function on $[0,1]\times[0,\varepsilon_0]$ for ε_0 sufficiently small, $H_2(1,\varepsilon) = 0$ and $H_2 \in C^1$ on $(0,1]\times[0,\varepsilon_0]$. Since the second term is negative and dominates the final term it follows that $h_2 < t$ on $(0,1)$ for ε sufficiently small. All other requirements for a supersub pair are also satisfied (e.g. one finds that $y_1/y_2 = 1 + \frac{2}{3}\varepsilon + O(\varepsilon^2)$). □

This supersub pair is sharp enough to determine the behaviour as $n \uparrow 2$.

Corollary 13 *The limit profile of* (u_n, L_n) *as* $n \uparrow 2$ *is* $u_n(x) = 1 - \frac{1}{2}(2-n)^{-1/2}x^2 + O(2-n)$ *and* $L_n = 2^{1/2}(2-n)^{1/4} + O((2-n)^{5/4})$.

Proof Let $n = 2-\varepsilon$. Let (y_1, y_2) be the supersub pair from Lemma 12. One finds that $y_i(t) = 2^{1/2}\varepsilon^{-1/2}(1-t) + O(\varepsilon^{1/2})$, $i = 1,2$ for $t \in (0,1)$ and

$$y_i^{-1/2} = 2^{-1/4}\varepsilon^{1/4}\frac{1}{\sqrt{1-t}} + \varepsilon^{5/4}F_i(t,\varepsilon), \qquad i = 1,2,$$

where $F_i(t,\varepsilon)$ is uniformly integrable on $[0,1]$ for small $\varepsilon > 0$. Since $y_2 \leqslant z \leqslant y_1$ it follows that

$$\begin{aligned} x &= \pm 2^{-1/4}\int_u^1 \frac{1}{\sqrt{z(s)}}\,ds = \pm 2^{-1/2}\varepsilon^{1/4}\int_u^1 \frac{1}{\sqrt{1-s}}\,ds + O\left(\varepsilon^{5/4}\right) \\ &= \pm 2^{1/2}\varepsilon^{1/4}\sqrt{1-u} + O\left(\varepsilon^{5/4}\right). \end{aligned}$$

Hence $L_n = 2^{1/2}\varepsilon^{1/4} + O(\varepsilon^{5/4})$ and $u(x) = 1 - 2\varepsilon^{-1/2}x^2 + O(\varepsilon)$ for $x \in (-L_n, L_n)$. □

3 Numerical methodology

Numerical solutions to the time-dependent problem (1.1) were obtained in Bowen & King [10]; here we describe the first simulations of the ordinary differential equation (1.2). The numerical self-similar solutions on which Figure 2 is based are obtained by solving the problem (1.2) on a half-domain $0 \leqslant x \leqslant 1$ (it is a simple matter to use the scaling invariance of (1.1) to scale L to 1) with symmetry conditions $v'(0) = v'''(0) = 0$. For $n < \frac{1}{2}$ equation (1.2) is then discretised using centred finite differences for the derivatives and an average for v^n; for a discussion of issues related to the type of average chosen see [10]. Choosing a uniform mesh $\{x_0 = 0, x_1, x_2, \ldots, x_N = 1\}$, equation (1.2) leads to a set of nonlinear equations for $v_i = v(x_i)$, $0 \leqslant i \leqslant N$; in practice we take $N = 10000$ (this large number of (uniformly spread) mesh points is chosen to ensure accurate results even in the singular limits $n \downarrow 0$ and $n \uparrow 2$). We fix the symmetry boundary conditions at $x = 0$ using ghost points, $v_{-1} = v_1$ and $v_{-2} = v_2$. At $x = 1$ we take $v_N = 0$ and $v_{N+1} = v_{N-1}$, implying a zero contact angle; alternatively we could impose the quadratic behaviour as given by (1.6) directly. The resulting system of $N + 1$ nonlinear equations is then solved by Newton iteration. For the numerical results presented above (see Figure 2) we start from $n = 0.49$ and then use continuation in n to decrease the value towards zero.

For $n > \frac{1}{2}$ it is an important numerical consideration that the second derivative of the interface behaviour blows up as $x \to 1$, cf. (1.6). To overcome the singular behaviour in the second derivative we define $v = f^{3/(n+1)}$ (a transformation which also enables the boundary layer behaviour for n close to 2 to be captured readily), whereby f satisfies

$$\left(f^2 f''' + \frac{3(2-n)}{n+1} f f' f'' + \frac{(2-n)(1-2n)}{(n+1)^2} (f')^3\right)' = \frac{n+1}{3n} f^{3/(n+1)}. \tag{3.1}$$

and from (1.6) we infer that $f(x) \sim C(1-x)$ as $x \to 1^-$ for some $C > 0$. The symmetry conditions at $x = 0$ are retained and we impose this linear behaviour on f at $x = 1$; this is achieved via the 'boundary condition' $f' \sim -f/(1-x)$, which is discretised using a ghost point. Equation (3.1) is discretised in a similar manner to (1.2) (again using centred differences) and the resulting set of nonlinear equations solved using Newton iteration with continuation as outlined above. The results of this process are also shown in Figure 2, with the two codes yielding consistent results as n approaches $\frac{1}{2}$ from below and above.

We can also determine the quantity $v^n v'''$ at $x = 1$ (corresponding to the flux leaving the domain through $x = 1$) numerically. Integrating (1.2) between $x = -1$ and $x = 1$ and using the symmetry of the solution (as provided by Theorem 1) yields

$$v^n \frac{d^3 v}{dx^3}\bigg|_{x=1} = \frac{1}{2n} \int_{-1}^{1} v \, dx \tag{3.2}$$

where the quantity on the right hand side is easily approximated from the numerical solution. Formal analysis implies

$$v^n v'''|_{x=1} \sim \frac{1}{3\sqrt{8\varepsilon}} \left(1 - \frac{1}{4}\varepsilon \ln \varepsilon - \frac{\varepsilon}{4}(1 - \ln 2)\right) \qquad \text{as} \quad n \uparrow 2, \tag{3.3}$$

with $\varepsilon = 2 - n$, where we have included corrections to the leading order result from Bowen & King [10]. The leading order term again follows rigorously from Theorem 1; we see from Figure 4 that the correction terms have a lesser effect than those for $||v||_\infty$ (cf. (1.8)). For the limit $n \downarrow 0$ one finds the asymptotic expression

$$v^n v'''|_{x=1} \sim \alpha^{n+1} \lambda_0^3 \left(n\lambda_0^4\right)^{-(n+1)/n} (\cosh \lambda_0 \sin \lambda_0 - \cos \lambda_0 \sinh \lambda_0) \qquad \text{as} \quad n \downarrow 0, \tag{3.4}$$

(with $\lambda_0 \approx 2.365$ and $\alpha \approx 0.5523$, as defined at the end of Section 1). We plot these approximations and the numerical result for (3.2) in Figure 4.

4 Power series solutions

The purpose of this section is to note briefly some power-series solutions which provide an alternative approach to the purely numerical one in solving the boundary-value

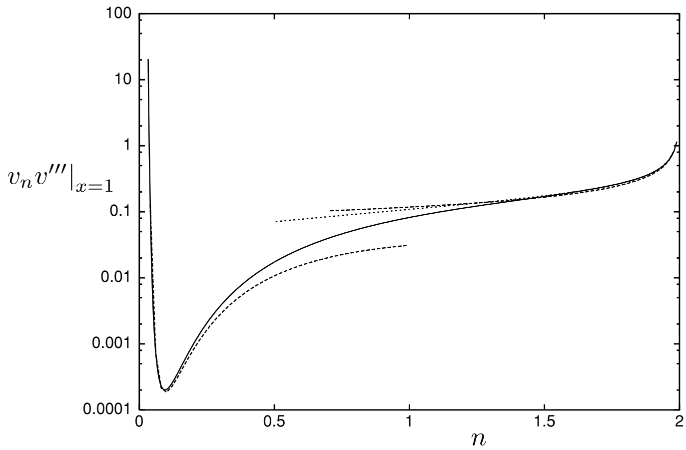

FIGURE 4. A comparison of numerically calculated values of the flux (solid line) with the asymptotics (leading order of (3.3) and (3.4); dashed lines) and the higher order approximation (3.3) (dotted line) for $0<n<2$.

problems outlined above; we record the procedure for completeness and because it is also applicable to other types of (similarity) solution which are in most other respects significantly more difficult to analyse. Encouragement for such an approach is provided by the rapid convergence of such series for the second-order problem (the porous medium equation, see King [13]). Besides, extending the asymptotic analysis of the limits $n \uparrow 2$ and $n \downarrow 0$ into the complex plane suggests that in the former case the only singularities are at $x = \pm 1$ (with the leading order solution having the power-series representation (1.7)), while in the latter case the singularities are located in the limit at values of x such that

$$\cosh \lambda_0 \cos(\lambda_0 x) = \cos \lambda_0 \cosh(\lambda_0 x),$$

which implies that the nearest complex singularities to $x = \pm 1$ are located at $x \approx \pm 2.42 \pm 1.57\,i$; those nearest to $x = -1$ thus have $|x+1| \approx 2.11$, so the singularity at $x = +1$ will (just) determine the radius of convergence of the power series about $x = -1$.

The approach is most readily implemented in the special case $n = -1$, it being shown in Bowen & King [10, § 6.1], that a first integral of (1.3) can then be obtained, leading to the third order problem

$$\frac{d^3 g}{dx^3} = \frac{1}{2}\left(g^2 - g_0^2\right) \tag{4.1}$$

where $g_x = u$ and g_0 is an unknown constant. Seeking a power series solution for equation (4.1) of the form $g(x) = \sum_{m=0}^{\infty} a_m (x+1)^{3m}$ yields the recurrence relation

$$(3m+3)(3m+2)(3m+1)a_{m+1} = \frac{1}{2}\sum_{k=0}^{m} a_k a_{m-k} \qquad m = 1, 2, \ldots \tag{4.2}$$

with $g_0^2 = a_0^2 - 12a_1$, where a_0 and a_1 are the free parameters in the local expansion, but can be fixed by imposing $g = g'' = 0$ at $x = 0$ provided the radius of convergence is greater than unity (as seems to be the case) and in this way we find $a_0 \approx -41.35276182$,

$a_1 \approx 57.00169701$; we estimate numerically that the radius of convergence is close (if not equal) to two and the power series expansion given by (4.2) therefore provides a representation of the solution on the whole domain $x = -1$ to $x = +1$ and converges extremely rapidly to the left of the symmetry boundary $x = 0$ (in practice only five terms are needed for 5%, and eight for 1%, accuracy). Somewhat remarkably, the relevant complex plane singularity structure in this case can in effect be determined *a priori*, as follows. Complex plane singularities to (4.1) have

$$g \sim 120/(x_c - x)^3 \qquad \text{as} \qquad x \to x_c; \tag{4.3}$$

we anticipate there will be such singularities on the real axis, having $|x| > 1$, and also on the imaginary axis (on which g is purely imaginary), the latter being closer to the origin than the former. Moreover, we have the symmetries

$$g(x) = -g(-x), \qquad g(x+1) = g\big(e^{2\pi i/3}(x+1)\big), \qquad g(x-1) = g\big(e^{2\pi i/3}(x-1)\big). \tag{4.4}$$

The most 'economical' way in which the three nearest singularities to $x = -1$ can be arranged is therefore to have one on the real axis and two on the imaginary axis, in which case the second of (4.4) implies that the singularities are located at

$$x_c = -3, \qquad x_c = \pm i\sqrt{3}. \tag{4.5}$$

The radius of convergence is thus two (while the singularities do not have to be arranged in the simplest fashion compatible with (4.4), the numerical evidence that they do so (obtained by solving (4.2)) is overwhelming). Determining a_m for large m from the superposition of (4.3) over the three nearest singularities x_c, given by (4.5), implies

$$a_m \sim -\frac{45}{2}(-1)^m \frac{(3m+2)(3m+1)}{2^{3m}} \qquad \text{as} \qquad m \to \infty; \tag{4.6}$$

the correction terms to (4.5) due to g_0 are a factor $(3m)^6$ smaller, while those due to the eigenmodes determined by linearising (4.1) about (4.3), namely $(x_c - x)^{(7\pm\sqrt{71}i)/2}$, are smaller in size by a factor $(3m)^{13/2}$. Moreover, by the first of (4.4), the next nearest singularities may have $|x+1| = 4$; whether this is true or not, we may expect (4.6) to provide an extremely accurate representation, and this is confirmed by comparison with the numerical results giving ten significant figures correctly (commensurate with the accuracy of a_0 and a_1) by the tenth term in the series. By combining the solutions to (4.2) for smaller m with (4.6), we may thus obtain a series solution which is exceptionally accurate (and is consistent with the numerical results in Bowen & King [10]).

For other n, we formulate the problem in the integral equation form

$$\frac{d^3u}{dx^3} = -\frac{J}{u^n} + \frac{1}{u^n}\int_{-1}^{x} u(x')\,dx', \tag{4.7}$$

where the constant J (the flux through $x = -1$, cf. (1.5)) must be determined as part of the solution. We illustrate the procedure by considering the range $n < \frac{1}{2}$. Iterating in (4.7) starting from the first of (1.6) yields

$$
\begin{aligned}
u \sim\ & C(x+1)^2 - \frac{J}{C^n\alpha}(x+1)^{3-2n} - \frac{nJ^2}{C^{2n+1}\alpha\beta}(x+1)^{4-4n} \\
& - \frac{nJ^3(2n\alpha+\beta(n+1))}{2\alpha^2\beta\delta C^{3n+2}}(x+1)^{5-6n} \\
& - \frac{J^4 n(3n\alpha[(\beta+2\delta)(n+1)+2n\alpha]+\beta\delta(n+1)(n+2))}{6\alpha^3\beta\delta\theta C^{4n+3}}(x+1)^{6-8n} \\
& + \frac{1}{3\gamma C^{n-1}}(x+1)^{6-2n},
\end{aligned}
\tag{4.8}
$$

where $\alpha=(1-2n)(2-2n)(3-2n)$, $\beta=(2-4n)(3-4n)(4-4n)$, $\gamma=(4-2n)(5-2n)(6-2n)$, $\delta=(3-6n)(4-6n)(5-6n)$ and $\theta=(4-8n)(5-8n)(6-8n)$; the last term in (4.8) is the first contribution from the integral in (4.7) – where it appears in the expansion depends on the value of n. This expression furnishes a very precise representation of the local singularity at $x=-1$ (and in this sense is preferable to the Taylor series expansion about $x=0$, which would furnish an alternative series representation); approximate values for C and J can be determined by imposing $u_x=u_{xxx}=0$ on $x=0$. Obviously, higher order terms can also be constructed, but we shall not pursue the approach further, it being clear from Figures 2–4 that the asymptotics in n in any case provide adequate analytical approximations for a wide range of exponents. We note, however, that in the special cases $n=(k-4)/2(k-1)$, $k=2,3,4,\ldots$, the contributions from the two terms on the right-hand side of (4.7) involve the same sequence of powers of $(x+1)$, giving

$$
u(x)=(x+1)^2\sum_{m=0}^{\infty} b_m(x+1)^{3m/(k-1)}, \tag{4.9}
$$

generalising the $n=-1$ ($k=2$) result above; the integral in (4.7) first contributes in the term b_k, so in practice at least this number of terms is needed. Finally, the above analysis can be carried out in an analogous manner for $1/2<n<2$. We find for $k(4n-5+\sqrt{1+20n-8n^2})=2(n+4)$, $k=3,4,5,\ldots$, that

$$
u(x)=(x+1)^{3/(n+1)}\sum_{m=0}^{\infty} b_m(x+1)^{m(n+4)/k(n+1)}. \tag{4.10}
$$

5 Discussion

An important open problem in the study of separable solutions is the role they play in the description of the large time (or extinction) behaviour of solutions of (1.1). This is a more general problem faced in relation to self-similar solutions of the thin film equation. Higher-order partial differential equations such as the thin film equation do not exhibit a comparison principle so that many of the methods used for second-order degenerate equations (e.g. the porous medium equation) cannot be applied. This issue has so far only been resolved for the special value $n=1$ in the case of source type solutions [11]. For $n=1$, one may hope to apply this method to the separable solution.

The separable solutions studied in this paper are another step towards a global study of all self-similar solutions to the thin film equation. The main challenge concerns the so-called self-similar solutions of the second kind, which refers to situations in which the exponents cannot be determined *a priori* via dimensional analysis and conservation laws. Such situations arise for example in the dipole problem when $n \neq 1$ (cf. Bowen *et al.* [9]) and for the hole-filling (focussing) problem.

Finally, multidimensional (even radially symmetric) generalisations of (1.1) await investigation (cf. Ferreira & Bernis [12] for source type solutions). Also, the method in this paper can be generalised to equations of the form $h_t = -(h^k(h^l(h^m h_x)_x)_x)_x$ with $k, l, m \in \mathbb{R}$ (cf. King [15]). For example, when $k = l = 0$ then the resulting ordinary differential equation has a variational structure and it is expected that there exists a separable solution if and only if $m > -1$, or in more dimensions (with $h_t = -\Delta^2(h^{m+1})/(m+1)$) when $m > \max\{-1, -8/(N+4)\}$ where N is the dimension, at least in a star-shaped domain (the local behaviour near the boundary requires $m > -1$, while the other lower bound (for $N > 4$) follows from a (global) Pohozaev type identity). This is comparable to the situation with the porous medium equation $h_t = \nabla \cdot (h^m \nabla h)$, for which the critical exponent is $m = \max\{-1, -4/(N+2)\}$. In contrast, for the thin film equation $h_t = -\nabla \cdot (h^n \nabla \Delta h)$, we expect separable solutions to exist in any dimension when $n < 2$, it being noteworthy that in this case the local behaviour near the boundary gives an upper bound rather than a lower one.

Acknowledgements

The first three authors gratefully acknowledge the support of the EPSRC as well as the RTN project Fronts-Singularities. JB is also supported by NWO.

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